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Affine images of compact convex sets and maximal measures ☆

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Abstract

Let $\varphi : X \to Y$ be an affine continuous mapping of a compact convex set X onto a compact convex set Y. We show that the induced mapping $\varphi_\sharp$ need not map maximal measures on X to maximal measures on Y even in case φ maps extreme points of X to extreme points of Y. This disproves Théorème 6 of [S. Teleman, Sur les mesures maximales, C. R. Acad. Sci. Paris Sér. I Math. 318 (6) (1994) 525–528]. We prove the statement of Théorème 6 under an additional assumption that $\operatorname{ext} Y$ is Lindelöf or Y is a simplex. We also show that under either of these two conditions injectivity of φ on $\operatorname{ext} X$ implies injectivity of $\varphi_\sharp$ on maximal measures. A couple of examples illustrate the results.

Résumé

Soit $\varphi : X \to Y$ une application affine et continue d'un compact convexe X sur un compact convexe Y. Nous montrons que l'image d'une mesure maximale par l'application induite $\varphi_\sharp$ n'est pas nécessairement une mesure maximale, même pas, si les images des points extrémaux sont des points extrémaux. Ceci réfute Théorème 6 dans [S. Teleman, Sur les mesures maximales, C. R. Acad. Sci. Paris Sér. I Math. 318 (6) (1994) 525–528]. Nous prouvons l'énoncé de ce théorème sous l'hypothèse supplémentaire que $\operatorname{ext} Y$ est Lindelöf ou Y est un simplex. En plus, nous démontrons que, en supposant l'une ou l'autre de ces deux propriétés, l'injectivité de φ sur $\operatorname{ext} X$ implique l'injectivité de φ pour les mesures maximales. Quelques exemples explicitent les résultats.

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1. Introduction

All topological spaces are considered to be Hausdorff. If X is a compact convex subset of a real locally convex space, we write $\operatorname{ext} X$ for the set of *extreme points* of X and $\mathcal{M}^1_{\max}(X)$ for the set of all *maximal probability Radon measures* on X (see [1, Chapter I, §3], we also refer the reader to [6, Chapter 6], [10, Sections 1–3], [2, Chapter 1], [15] and [13, Chapter 7]). If $\varphi : X \to Y$ is a continuous mapping of a compact space X to a compact set Y, it induces a continuous mapping $\varphi_\sharp : \mathcal{M}^1(X) \to \mathcal{M}^1(Y)$ from the set of all probability Radon measures on X to the set of all probability Radon measures on Y by the formula $\varphi_\sharp \mu = \mu \circ \varphi^{-1}$ (see [11, Theorem 418I]). The induced mapping $\varphi_\sharp$ is surjective if φ is surjective.

For any $\mu \in \mathcal{M}^1(X)$ we write $r(\mu)$ for the *barycenter* of μ (see [1, Chapter I, §2]). If $x \in X$, we write $\mathcal{M}_x$ for the set of all measures $\mu \in \mathcal{M}^1(X)$ satisfying $r(\mu) = x$. We recall that a set $F \subset X$ is *extremal* if $x, y \in F$ whenever $x, y \in X$, $\alpha \in (0, 1)$ and $\alpha x + (1 - \alpha)y \in F$. It is a *face* if F is a convex extremal set. We also mention the well-known fact that $\operatorname{ext} F = F \cap \operatorname{ext} X$ for any face F.

Let $\varphi : X \to Y$ be a continuous affine mapping of a compact convex set X to a compact convex set Y. If $\varphi : X \to Y$ is surjective, it is easy to see that $\varphi(\operatorname{ext} X) \supset \operatorname{ext} Y$ and $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \supset \mathcal{M}^1_{\max}(Y)$. In order to ensure the reverse inclusion $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$, it is necessary to assume that $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$. This observation prompts the following two questions.

Question. Let $\varphi : X \to Y$ be a continuous affine mapping of a compact convex X to a compact convex set Y.

(1) If $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$, does it imply that $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$?
(2) If $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$, does it imply that $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$?

If Y is a simplex (see [1, Chapter II, §3]), both questions were answered affirmatively in [8, Corollaries 2 and 3]. For X and Y being simplices, the result can be found in [7, Lemma 6] and [12, Theorem 1]. It is claimed in [18, Théorème 6] without a proof that Question (1) has the affirmative answer without any restrictions. The author also suggests to study Question (2) in [18, Conjecture].

Unfortunately, the answer to Question (1) is in general negative as the following example shows (see also [3, Example 1]).

Example 1.1. There exists a continuous affine surjection φ of a simplex X onto a compact convex set Y and a measure $\mu \in \mathcal{M}^1_{\max}(X)$ such that

- $\varphi(\operatorname{ext} X) = \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$,
- $\varphi_\sharp \mu \notin \mathcal{M}^1_{\max}(Y)$.

Nevertheless, we prove in Theorem 1.2 that the answer to both questions is positive if we assume that $\operatorname{ext} Y$ is a Lindelöf space (see [9, Section 3.8]).

Theorem 1.2. *Let $\varphi : X \to Y$ be a continuous affine map of a compact convex set X to a compact convex set Y and let* $\operatorname{ext} Y$ *be a Lindelöf space.*

(a) *Then the following assertions are equivalent*:
  (i) $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$,
  (ii) $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$.
(b) *Further, the following assertions are equivalent*:
  (i′) $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ *and φ is injective on* $\operatorname{ext} X$,
  (ii′) $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$ *and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$.*

We also provide in Theorem 1.3(a) a slightly different proof of [8, Corollary 2]. The case of injectivity is described in Theorem 1.3(b), where the proof is based upon the results of E.A. Reznichenko from [16]. We indicate in Remark 2.4 an alternative proof of this assertion that uses a notion of induced measures on the set of extreme points, which is a technique developed by S. Teleman and C.J.K. Batty in [19] and [4].

Theorem 1.3. *Let $\varphi : X \to Y$ be a continuous affine map of a compact convex set X to a simplex Y.*

(a) *Then the following assertions are equivalent*:
  (i) $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$,
  (ii) $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$,
  (iii) *$\varphi(F)$ is a face for each closed face $F \subset X$,*
  (iv) *$\varphi(F)$ is a closed extremal set for each closed extremal $F \subset X$.*
(b) *Further, the following assertions are equivalent*:
  (i′) $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ *and φ is injective on* $\operatorname{ext} X$,
  (ii′) $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$ *and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$,*
  (iii′) *φ is a homeomorphism onto $\varphi(X)$.*

The following example shows that Theorem 1.3(b) need not hold if we omit the condition imposed on Y.

Example 1.4. There exists a continuous affine surjection φ of a metrizable simplex X onto a compact convex set Y such that

- φ is injective on $\operatorname{ext} X$,
- $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$ and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$,
- φ is not injective on X.

Our last example shows that even if $\varphi_\sharp$ maps maximal measures to maximal measures and φ is injective on $\operatorname{ext} X$, the induced mapping need not be injective on $\mathcal{M}^1_{\max}(X)$.

Example 1.5. There exists a continuous affine surjection φ of a simplex X onto a compact convex set Y such that

- φ is injective on $\operatorname{ext} X$,
- $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$,
- $\varphi_\sharp$ is not injective on $\mathcal{M}^1_{\max}(X)$.

2. Proofs of the positive results

If $f : X \to \mathbb{R}$ is a function on a compact convex set X, we recall the definition from [1, p. 4] of the *upper envelope* f^* of f defined as

$$f^*(x) = \inf\{h(x)\colon\ h \geqslant f,\ h \text{ continuous affine on } X\}, \quad x \in X.$$

Before embarking on the proof of the main theorems, we need a couple of auxiliary results.

Proposition 2.1. *Let f, g, be upper semicontinuous real functions on X such that f is concave, g is convex and $f \geqslant g$ on* $\operatorname{ext} X$. *Then $f \geqslant g$ on X.*

Proof. Given f and g as in the premise, let x be a point of X. We fix $\varepsilon > 0$ and use [1, Corollary I.1.3] to find a concave continuous function f' such that $f' \geqslant f$ and $f(x) \geqslant f'(x) - \varepsilon$.

Then $f' - g$ is a lower semicontinuous concave function on X such that $f' - g \geqslant 0$ on $\operatorname{ext} X$. According to Bauer's minimum principle [1, Theorem I.5.3], $f' - g \geqslant 0$ on X. Thus

$$g(x) \leqslant f'(x) \leqslant f(x) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we are done. □

Proposition 2.2. *Let* $\operatorname{ext} X$ *be Lindelöf and $\mu \in \mathcal{M}^1(X)$. Then the following assertions are equivalent*:

(i) $\mu \in \mathcal{M}^1_{\max}(X)$,
(ii) $\mu_*(X \setminus \operatorname{ext} X) = 0$ (*here μ_* stands for the inner measure induced by μ*).

Proof. Let $\mu \in \mathcal{M}^1_{\max}(X)$ be given and $F \subset X \setminus \operatorname{ext} X$ be an arbitrary closed set. For any point $x \in \operatorname{ext} X$ we can find a cozero set U_x such that $x \in U_x \subset X \setminus F$. (We recall that a subset of a normal space is cozero if and only if it is an open F_σ set, see [9, p. 42].) By the Lindelöf property of $\operatorname{ext} X$, there exists a cozero set U such that

$$\operatorname{ext} X \subset U \subset X \setminus F.$$

According to [6, Theorem 27.11], $\mu(U) = 1$ and hence $\mu(F) = 0$. Thus $\mu_*(X \setminus \operatorname{ext} X) = 0$ and (i) ⇒ (ii).

For the proof of (ii) ⇒ (i), let μ satisfy (ii). For any continuous function f on X, [1, p. 32] yields

$$\operatorname{ext} X \subset \{x \in X\colon\ f^*(x) = f(x)\}.$$

Hence $\mu(\{x \in X\colon\ f^*(x) = f(x)\}) = 1$ and $\mu(f^*) = \mu(f)$. By [1, Proposition I.4.5], $\mu \in \mathcal{M}^1_{\max}(X)$. □

Proof of Theorem 1.2. For the proof of (a) we first notice that the implications (ii) ⇒ (i) and (ii′) ⇒ (i′) are obvious. We start the proof of the converse implications by showing (i) ⇒ (ii). To this end, let $\mu \in \mathcal{M}^1_{\max}(X)$ be given. We fix an arbitrary closed set $F \subset Y \setminus \operatorname{ext} Y$. Since $\operatorname{ext} Y$ is Lindelöf, there exists a countable family of cozero sets $\{U_n\colon\ n \in \mathbb{N}\}$ in Y such that

$$\operatorname{ext} Y \subset \bigcup_{n=1}^{\infty} U_n \subset Y \setminus F.$$

Then $G = \varphi^{-1}(\bigcup_{n=1}^{\infty} U_n)$ is an F_σ set. By the assumptions, $\operatorname{ext} X \subset G$ and hence $\mu(G) = 1$. Thus

$$(\varphi_\sharp \mu)\left(\bigcup_{n=1}^{\infty} U_n\right) = \mu(G) = 1,$$

and hence $\mu(F) = 0$.

Thus $(\varphi_\sharp \mu)_*(Y \setminus \operatorname{ext} Y) = 0$, and $\varphi_\sharp \mu \in \mathcal{M}^1_{\max}(Y)$ by virtue of Proposition 2.2.

We proceed with the proof of (i′) ⇒ (ii′). We start by proving

$$\varphi(X \setminus \operatorname{ext} X) \subset Y \setminus \operatorname{ext} Y. \tag{1}$$

Indeed, given $y \in \operatorname{ext} Y \cap \varphi(X)$, the set $\varphi^{-1}(y)$ is a closed face. Since

$$\varphi^{-1}(y) = \overline{\operatorname{co}}\big(\operatorname{ext} \varphi^{-1}(y)\big) = \overline{\operatorname{co}}\big(\varphi^{-1}(y) \cap \operatorname{ext} X\big),$$

the assumption yields that $\varphi^{-1}(y)$ is a singleton. Hence (1) follows.

Let $\mu \in \mathcal{M}^1_{\max}(X)$ be given. For any set $F \subset X \setminus \operatorname{ext} X$, inclusion (1) gives

$$\varphi(F) \subset Y \setminus \operatorname{ext} Y.$$

This along with Proposition 2.2 and the first part of the proof yields

$$(\varphi_\sharp \mu)\big(\varphi(F)\big) = 0, \quad F \subset X \setminus \operatorname{ext} X \text{ closed}.$$

Hence

$$\mu(F) \leqslant \mu\big(\varphi^{-1}\big(\varphi(F)\big)\big) = (\varphi_\sharp \mu)\big(\varphi(F)\big) = 0, \quad F \subset X \setminus \operatorname{ext} X \text{ closed},$$

and thus

$$\mu\big(\varphi^{-1}\big(\varphi(F)\big)\big) = \mu(F), \quad F \subset X \text{ closed}. \tag{2}$$

If $\mu, \nu \in \mathcal{M}^1_{\max}(X)$ are measures with $\varphi_\sharp \mu = \varphi_\sharp \nu$, then (2) yields

$$\mu(F) = \mu\big(\varphi^{-1}\big(\varphi(F)\big)\big) = (\varphi_\sharp \mu)\big(\varphi(F)\big) = (\varphi_\sharp \nu)\big(\varphi(F)\big) = \nu\big(\varphi^{-1}\big(\varphi(F)\big)\big) = \nu(F)$$

for any closed $F \subset X$. Hence $\mu = \nu$ and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$. □

Remark 2.3. It can be easily verified that the mapping $\varphi : X \to Y$ is a homeomorphism of $\operatorname{ext} X$ onto $\varphi(\operatorname{ext} X)$ if $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$.

Indeed, since

$$\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y \quad \text{and} \quad \varphi(X \setminus \operatorname{ext} X) \subset Y \setminus \operatorname{ext} Y,$$

it is not difficult to realize that $\varphi(F \cap \operatorname{ext} X) = \varphi(F) \cap \operatorname{ext} Y$ for any $F \subset X$. Hence $\varphi : \operatorname{ext} X \to \operatorname{ext} Y$ is a closed mapping, and thus a homeomorphism on $\operatorname{ext} X$.

Hence we obtain that $\operatorname{ext} X$ is a Lindelöf space if $\operatorname{ext} Y$ is Lindelöf and φ as above.

Proof of Theorem 1.3. For the proof of (a), we first verify (i) ⇒ (ii). To this end, let μ be a maximal probability measure on X. To show that $\varphi_\sharp \mu$ is maximal on Y, we use Mokobodzki's maximality test [1, Proposition I.4.5].

Let g be a convex continuous function on Y. Since Y is a simplex, g^* is an affine function (see [1, Theorem II.3.7]). By the assumption and [1, Proposition I.4.1],

$$g^* \circ \varphi = (g \circ \varphi)^* \quad \text{on } \operatorname{ext} X.$$

By Proposition 2.1, $g^* \circ \varphi \leqslant (g \circ \varphi)^*$ on X.

On the other hand, given $x \in X$, there exists a measure $\lambda \in \mathcal{M}_x$ such that $\lambda(g \circ \varphi) = (g \circ \varphi)^*(x)$ (see [1, Proposition I.3.5]). Then $\varphi_\sharp \lambda \in \mathcal{M}_{\varphi(x)}$ and

$$(g \circ \varphi)^*(x) = \lambda(g \circ \varphi) = (\varphi_\sharp \lambda)(g) \leqslant g^*\big(\varphi(x)\big).$$

Hence $g^* \circ \varphi = (g \circ \varphi)^*$ on X.

Thus the equality

$$(\varphi_\sharp \mu)(g) = \mu(g \circ \varphi) = \mu\big((g \circ \varphi)^*\big) = \mu(g^* \circ \varphi) = (\varphi_\sharp \mu)(g^*)$$

shows that $\varphi_\sharp \mu$ is a maximal measure on Y.

We proceed with the proof by showing (ii) $\Rightarrow$ (iii). Let $F \subset X$ be a closed face. Since $\varphi(F)$ is obviously convex, we need to check its extremality.

Let $\nu \in \mathcal{M}^1_{\max}(Y)$ satisfy $r(\nu) \in \varphi(F)$. We find a point $x \in F$ with $\varphi(x) = r(\nu)$ and select a measure $\mu \in \mathcal{M}^1_{\max}(X)$ such that $r(\mu) = x$. Since F is a closed face, $\mu \in \mathcal{M}^1(F)$. Then $\varphi_\sharp \mu$ is supported by $\varphi(F)$ and by the assumption, $\varphi_\sharp \mu$ is maximal. Since

$$r(\varphi_\sharp \mu) = \varphi\big(r(\mu)\big) = r(\nu)$$

and Y is a simplex, $\varphi_\sharp \mu = \nu$. Hence $\nu \in \mathcal{M}^1(\varphi(F))$.

Let now an arbitrary $\nu' \in \mathcal{M}^1(Y)$ satisfy $r(\nu') \in \varphi(F)$. We find a maximal measure $\nu \in \mathcal{M}^1_{\max}(Y)$ such that $\nu' \preceq \nu$ (here $\preceq$ is the Choquet ordering, see [1, Chapter I, §3] and [1, Lemma I.4.7]). Since $r(\nu) = r(\nu')$, ν is supported by $\varphi(F)$ according to the paragraph above. Since it is easy to see that $\operatorname{spt} \nu' \subset \overline{\operatorname{co}} \operatorname{spt} \nu$, the measure ν' is supported by $\varphi(F)$ as well. Thus $\varphi(F)$ is a face as needed.

Since a closed set is extremal if and only if it is a union of closed faces (see [14, §4, Theorem 7]), we get (iii) $\Rightarrow$ (iv). We proceed to the proof of (iv) $\Rightarrow$ (i). But this is immediate, because a set $\{x\}$ is extremal if and only if $x \in \operatorname{ext} X$. This concludes the proof of (a).

We start the proof of (b) by showing (i′) $\Rightarrow$ (iii′). We know from the part (a) that $\varphi(X)$ is a face of Y and hence a simplex. Since $\operatorname{ext} \varphi(X) = \varphi(X) \cap \operatorname{ext} Y$, we may assume from now on that φ is a surjective mapping onto a simplex Y.

Thus we may use [16, Proposition 1.6] to get that φ is a simplicial map, that is, the function

$$\tilde{a}(y) = \inf\big\{a(x):\ x \in \varphi^{-1}(y)\big\}, \quad y \in Y,$$

is affine on Y for any continuous affine function a on X (see [16, Definition 1.3]). Since φ is injective on $\operatorname{ext} X$, [16, Theorem 1.5] yields that φ is a homeomorphism.

Since the remaining implications are obvious, the proof is finished. □

Remark 2.4. We remark that Theorem 1.3(b) can be deduced from results of S. Teleman and C.J.K. Batty on maximal measures.

For the proof of (i′) $\Rightarrow$ (ii′) we realize that $F = \varphi^{-1}(\varphi(F))$ for any closed face $F \subset X$ and hence also for any closed extremal set $F \subset X$. It is shown in [4, Section 6] or in [19, Theorem 5.2] and [20, Theorem 6] that

$$\mu(B) = \sup\big\{\mu(F):\ F \subset B \text{ is closed extremal}\big\}, \quad B \subset X \text{ Baire},$$

for any measure $\mu \in \mathcal{M}^1_{\max}(X)$. From this fact we get that $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$.

To verify (ii′) $\Rightarrow$ (iii′), it is enough to check injectivity of φ on X. Let $x_1, x_2 \in X$ satisfy $y = \varphi(x_1) = \varphi(x_2)$. For $i = 1, 2$, we find a maximal measure $\mu_i \in \mathcal{M}_{x_i}$. Then the measure

$\varphi_\sharp\mu_i \in \mathcal{M}_y$, $i = 1, 2$, and thus $\varphi_\sharp\mu_1 = \varphi_\sharp\mu_2$ (we remind that Y is a simplex). By the assumption, $\mu_1 = \mu_2$ and thus $x_1 = x_2$.

Obviously, (iii′) ⇒ (i′) which finishes this remark.

3. Construction of examples

All the constructions are based upon the notion of a *function space* $\mathcal{H}$, which is a subspace of the space $\mathcal{C}(K)$ of all continuous functions on a compact space K such that $\mathcal{H}$ contains constant functions and separates points of K. Then the *state space*

$$X = \{\xi \in \mathcal{H}^*:\ \xi \geqslant 0,\ \xi(1) = 1\}$$

endowed with the weak* topology is a convex compact set that inherits many properties from $\mathcal{H}$. The mapping $\phi : K \to X$, where $\phi(x)$ is the evaluation mapping at a point $x \in K$, is a homeomorphic embedding. (We refer the reader to [15, Chapter 6], [6, Chapter 6, §29] and [17] for a detailed information on the issue.)

Construction of Example 1.1. Let $K_1 = [0, 1] \times \{-1, 0, 1\}$ with the "porcupine" topology (see [5, Section VII] or [1, Proposition I.4.15]) and let K_2 be the quotient of K_1 where all points of $[0, 1] \times \{0\}$ are identified with the point $(0, 0)$ (see [9, Section 2.4]). We write $q : K_1 \to K_2$ for the quotient mapping and take

$$\mathcal{H}_1 = \{f \in \mathcal{C}(K_1):\ 2f((t, 0)) = f((t, -1)) + f((t, 1)),\ t \in [0, 1]\} \quad \text{and}$$
$$\mathcal{H}_2 = \{f \in \mathcal{C}(K_2):\ 2f((0, 0)) = f((t, -1)) + f((t, 1)),\ t \in [0, 1]\}.$$

Let X, Y be the state space of $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, and ϕ_1, ϕ_2 be the respective embeddings. Then $\operatorname{ext} X = \phi_1(K_1 \setminus ([0, 1] \times \{0\}))$ and $\operatorname{ext} Y = \phi_2(K_2 \setminus \{(0, 0)\})$. We denote by $\varphi : X \to Y$ the restriction of the adjoint operator $h \mapsto h \circ q$, $h \in \mathcal{H}_2$. Then X is a simplex and $\phi_\sharp\lambda \in \mathcal{M}^1(X)$ is maximal for any continuous measure $\lambda \in \mathcal{M}^1([0, 1] \times \{0\})$, even though $\phi_\sharp\lambda$ is supported by a compact set disjoint with $\operatorname{ext} X$ (see [1, Chapter I, §4, p. 42]). (We recall that λ is continuous if $\lambda(\{x\}) = 0$ for each $x \in X$.)

Then $\varphi(\operatorname{ext} X) = \operatorname{ext} Y$ and φ is even injective on $\operatorname{ext} X$. On the other hand, if λ is any continuous probability measure on $\phi_1([0, 1] \times \{0\})$, then λ is maximal on X, yet the measure $\varphi_\sharp\lambda$ equals the Dirac measure at the point $\phi_2((0, 0))$, and hence $\varphi_\sharp\lambda$ is not maximal. □

Construction of Example 1.4. Let $K_1 = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and K_2 be the quotient of K_1, if we identify y_2 with x_2. Again we denote by $q : K_1 \to K_2$ the quotient mapping. Let

$$\mathcal{H}_1 = \{f \in \mathcal{C}(K_1):\ 2f(x_2) = f(x_1) + f(x_3),\ 2f(y_2) = f(y_1) + f(y_3)\} \quad \text{and}$$
$$\mathcal{H}_2 = \{f \in \mathcal{C}(K_2):\ f(x_1) + f(x_3) = 2f(x_2) = f(y_1) + f(y_3)\}.$$

We take X, Y, ϕ_1, ϕ_2 and $\varphi : X \to Y$ as above. Then X is a simplex, $\operatorname{ext} X = \phi_1(K_1 \setminus \{x_2, y_2\})$, $\operatorname{ext} Y = \phi_2(K_2 \setminus \{x_2\})$, $\varphi : \operatorname{ext} X \to \operatorname{ext} Y$ is a bijection, yet φ is not injective on X. Obviously, $\varphi_\sharp$ maps injectively maximal measures to maximal measures. □

Construction of Example 1.5. Let $K_1 = [0, 1] \cup [2, 3] \times \{-1, 0, 1\}$ endowed again with the "porcupine" topology and let K_2 be the quotient of K_1 after identifying points $(t + 2, 0)$ with $(t, 0)$, $t \in [0, 1]$. Let

$$\mathcal{H}_1 = \bigl\{ f \in \mathcal{C}(K_1)\colon\ 2f((t+i,0)) = f((t+i,-1)) + f((t+i,1)),\ t \in [0,1],\ i = 0,2 \bigr\},$$
$$\mathcal{H}_2 = \bigl\{ f \in \mathcal{C}(K_2)\colon\ 2f((t,0)) = f((t+i,-1)) + f((t+i,1)),\ t \in [0,1],\ i = 0,2 \bigr\},$$

and let X, Y, ϕ_1, ϕ_2 and φ be as above.

Then

$$\operatorname{ext} X = \phi_1\bigl(K_1 \setminus \bigl([0,1] \cup [2,3] \times \{0\}\bigr)\bigr), \qquad \operatorname{ext} Y = \phi_2\bigl(K_2 \setminus \bigl([0,1] \times \{0\}\bigr)\bigr),$$

and φ maps injectively $\operatorname{ext} X$ onto $\operatorname{ext} Y$.

We claim that $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$. Indeed, a probability measure λ is maximal on X if and only if $\lambda = (\phi_1)_\sharp \mu$ for some measure $\mu \in \mathcal{M}^1(K_1)$ that is continuous on $[0,1] \cup [2,3] \times \{0\}$. Similarly, any maximal probability measure on Y is of the form $(\phi_2)_\sharp \mu$ for some measure $\mu \in \mathcal{M}^1(K_2)$ that is continuous on $[0,1] \times \{0\}$. From these observations the claim follows.

Finally, if we take the Lebesgue measure λ_1 on $[0,1] \times \{0\}$ and λ_2 on $[2,3] \times \{0\}$, then

$$\varphi_\sharp\bigl((\phi_1)_\sharp \lambda_1\bigr) = \varphi_\sharp\bigl((\phi_1)_\sharp \lambda_2\bigr).$$

Hence $\varphi_\sharp$ is not injective on $\mathcal{M}^1_{\max}(X)$. $\square$

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